Lecture 8

7.1: Integration by Parts

In Calculus I, you reversed the chain rule:

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\,g'(x)$$

and got u-substitution:

$$\int f'(g(x))\,g'(x)\,dx = f(g(x)) + C$$

$(u = g(x))$.

This semester, we reverse the other key differentiation rule: the product rule:

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$$

Integrating this gives

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

(The constant of integration from the left side is incorporated into the indefinite integrals on the right side.)

A slight rearrangement of terms gives:

Integration by Parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Writing $u=f(x)$ and $v=g(x)$, we have the more common formulation:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Incorporating limits of integration, we have

$$\boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}$$

This is THE BIGGEST GUN you have to attack integration problems. In fact, this tool is <u>so important</u>, it is heavily used in <u>current</u> research in analysis and (partial) differential equations! Before getting into tricks, let's do a simple example:

Ex: $\displaystyle\int_1^4 xe^{2x}\,dx$ $\left(\begin{array}{ll} u=x & dv=e^{2x}dx \\ du=dx & v=\frac{1}{2}e^{2x}\end{array}\right)$

$$=\frac{1}{2}xe^{2x}\Big|_1^4-\int_1^4\frac{1}{2}e^{2x}dx$$

$$=\left(2e^8-\frac{1}{2}e^2\right)-\left(\frac{1}{4}e^{2x}\right)\Big|_1^4=\left(2e^8-\frac{1}{2}e^2\right)-\left(\frac{1}{4}e^8-\frac{1}{4}e^2\right)$$

$$=\boxed{\frac{7}{4}e^8-\frac{1}{4}e^2}$$

Sometimes, it may be necessary to apply IBP two (or more!) times:

Ex: $\displaystyle\int x^2\cos x\,dx$ $\left(\begin{array}{ll} u=x^2 & dv=\cos x\,dx \\ du=2x\,dx & v=\sin x\end{array}\right)$

$$=x^2\sin x-\int 2x\sin x\,dx \quad \left(\begin{array}{ll} u=2x & dv=\sin x\,dx \\ du=2dx & v=-\cos x\end{array}\right)$$

$$=x^2\sin x-\left(-2x\cos x+\int 2\cos x\,dx\right)$$

$$=x^2\sin x+2x\cos x-\int 2\cos x\,dx$$

$$=\boxed{x^2\sin x+2x\cos x-2\sin x+C}$$

Another common trick is letting $dv = dx$:

Ex: (a) $\int \arctan(x)\,dx$ $\left(\begin{array}{ll} u = \arctan x & dv = dx \\ du = \frac{1}{1+x^2}dx & v = x \end{array}\right)$

$$= x\arctan x - \int \frac{x}{1+x^2}dx \quad \left(\begin{array}{l} u\text{-sub:} \\ u = 1+x^2 \\ du = 2x\,dx \end{array}\right)$$

$$= x\arctan x - \int \frac{1/2}{u}du = \boxed{x\arctan x - \tfrac{1}{2}\ln|1+x^2| + C}$$

(b) $\int \ln(x)\,dx$ $\left(\begin{array}{ll} u = \ln x & dv = dx \\ du = \frac{1}{x}dx & v = x \end{array}\right)$

$$= x\ln x - \int dx = \boxed{x\ln x - x + C}$$

Finally, another common trick involves the recurrence of an integral:

Ex: $\int e^x \sin x\,dx$ $\left(\begin{array}{ll} u = \sin x & dv = e^x dx \\ du = \cos x\,dx & v = e^x \end{array}\right)$

$$= e^x\sin x - \int e^x \cos x\,dx \quad \left(\begin{array}{ll} u = \cos x & dv = e^x dx \\ du = -\sin x\,dx & v = e^x \end{array}\right)$$

$$= e^x \sin x - \left(e^x\cos x + \int e^x \sin x\,dx\right)$$

notice we add this now

$$\Rightarrow 2\int e^x \sin x\,dx = e^x \sin x - e^x\cos x + C$$

$$\Rightarrow \boxed{\int e^x \sin x\,dx = \frac{1}{2}\left(e^x\sin x - e^x\cos x\right) + C}$$

Since IBP is your most powerful tool, you should use it only when other methods won't work

1) Is there a formula for the integral?

e.g. $\int x^n\,dx$, $\int \cos x\,dx$, $\int e^{ax}\,dx$, etc. ...

2) Can the integral be done with a u-substitution?

e.g. $\int x e^{x^2}\,dx$, $\int \frac{\ln x}{x}\,dx$, $\int \frac{3x}{1+x^2}\,dx$, etc. ...

3) If you do use IBP, there is an acronym that can help you choose u & dv. Let u be whatever shows up first, and dv be the rest.

LIATE

- **L**ogarithm
- **I**nverse Trig.
- **A**lgebraic
- **T**rigonometric
- **E**xponential

Sometimes these are switched

The philosophy here is you want u to be easier to differentiate and dv to be easier to integrate.

Integration by parts can also be used to obtain the following reduction formulas: $(m \geq 2)$

$$\int \sin^m x\,dx \qquad \left(\begin{array}{ll} u = \sin^{m-1} x & dv = \sin x\,dx \\ du = (m-1)\sin^{m-2} x \cos x\,dx & v = -\cos x \end{array} \right)$$

$$\|$$

$$-\sin^{m-1} x \cos x + (m-1)\int \sin^{m-2} x \cos^2 x\,dx$$

$$= -\sin^{m-1} x \cos x + (m-1)\int \sin^{m-2} x\,(1-\sin^2 x)\,dx$$

$$= -\sin^{m-1} x \cos x + (m-1)\left(\int \sin^{m-2} x\,dx - \int \sin^m x\,dx\right)$$

$$\Rightarrow m\int \sin^m x\,dx = -\sin^{m-1} x \cos x + (m-1)\int \sin^{m-2} x\,dx$$

$$\Rightarrow \boxed{\int \sin^m x\,dx = \frac{-1}{m}\left[\sin^{m-1} x \cos x - (m-1)\int \sin^{m-2} x\,dx\right]}$$

$$\int \cos^m x\,dx \qquad \left(\begin{array}{ll} u = \cos^{m-1} x & dv = \cos x\,dx \\ du = -(m-1)\cos^{m-2} x \sin x\,dx & v = \sin x \end{array} \right)$$

$$\|$$

$$\cos^{m-1} x \sin x + (m-1)\int \cos^{m-2} x \sin^2 x\,dx$$

$$= \cos^{m-1} x \sin x + (m-1)\int \cos^{m-2} x\,(1-\cos^2 x)\,dx$$

$$= \cos^{m-1} x \sin x + (m-1)\left(\int \cos^{m-2} x\,dx - \int \cos^m x\,dx\right)$$

$$\Rightarrow m\int \cos^m x\,dx = \cos^{m-1} x \sin x + (m-1)\int \cos^{m-2} x\,dx$$

$$\Rightarrow \boxed{\int \cos^m x\,dx = \frac{1}{m}\left[\cos^{m-1} x \sin x + (m-1)\int \cos^{m-2} x\,dx\right]}$$